TEMPERATURE WAVES IN A CYLINDER

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We have derived the formulas for the determination of the rate of temperature-wave propagation and for the propagation of a heat-flow wave in a uniform isotropic cylinder. We prove that these rates vary.

The solution of the differential equation

$$\frac{\partial t(r,\ \tau)}{\partial \tau}=a\left[\frac{\partial^2 t(r,\ \tau)}{\partial r^2}+\frac{1}{r}\ \frac{\partial t(r,\ \tau)}{\partial r}\right],\qquad 0\leqslant r\leqslant R,\ \tau\geqslant 0 \tag{1}$$

for the boundary conditions

$$\left[\frac{\partial t(r,\ \tau)}{\partial r}\right]_{r=0}=0, \tag{2}$$

$$t(R,\ \tau)=t(R)\exp i\omega\tau \tag{3}$$

has the following form:

$$t(r,\ \tau)=t(0)I_0\left(r\sqrt{\frac{i\omega}{a}}\right)\exp i\omega\tau. \tag{4}$$

Having determined the amplitude of the temperature wave at the axis of the cylinder, after transformation, we can present the instantaneous values for the temperature and the heat flow by means of the following formulas:

$$t(r,\ \tau)=t(R)\frac{M_0\left(r\sqrt{\frac{\omega}{a}}\right)}{M_0\left(R\sqrt{\frac{\omega}{a}}\right)}\exp i\left[\omega\tau+\theta_0\left(r\sqrt{\frac{\omega}{a}}\right)-\theta_0\left(R\sqrt{\frac{\omega}{a}}\right)\right], \tag{5}$$

$$q(r,\ \tau)=t(R)\lambda\sqrt{\frac{\omega}{a}}\ \frac{M_1\left(r\sqrt{\frac{\omega}{a}}\right)}{M_0\left(R\sqrt{\frac{\omega}{a}}\right)}\exp i\left[\omega\tau+\theta_1\left(r\sqrt{\frac{\omega}{a}}\right)-\theta_0\left(R\sqrt{\frac{\omega}{a}}\right)-\frac{\pi}{4}\right], \tag{6}$$

where, in general form,

$$M_n(z)=\sqrt{\mathrm{ber}_n^2 z+\mathrm{bei}_n^2 z}, \tag{7}$$

$$\theta_n(z)=\mathrm{arctg}\,\frac{\mathrm{bei}_n z}{\mathrm{ber}_n z}, \tag{8}$$

$$I_n(z)=i^{-n}(\mathrm{ber}_n z+i\,\mathrm{bei}_n z),\quad n=1,\ 2\ldots \tag{9}$$

The moduli of (5) and (6), respectively, determine the real amplitude of the temperature wave and the real amplitude of the heat flow. The arguments of these expressions determine the phase shift, and the difference between the arguments of expressions (6) and (5) determines the mutual phase shift of these waves, i.e.,

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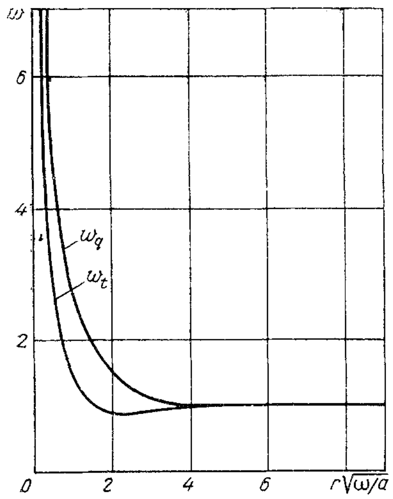

Fig. 1. Relative rate for the temperature and heat-flow waves.

$$\arg[q(r,\tau)] - \arg[t(r,\tau)] = \theta_1\left(r\sqrt{\frac{\omega}{a}}\right) - \theta_0\left(r\sqrt{\frac{\omega}{a}}\right) - \frac{\pi}{4}, \tag{10}$$

since

$$\lim_{z\to\infty}\left[\theta_n(z) - \theta_{n-1}(z)\right] = \frac{\pi}{2}, \tag{11}$$

$$\lim_{z\to 0}\left[\theta_n(z) - \theta_{n-1}(z)\right] = \frac{3}{4}\pi. \tag{12}$$

It follows from (10) that the heat-flow wave precedes the phase of the temperature wave by 1/8 of a period for $r\sqrt{\omega/a} \to \infty$ and by 1/4 of a period for $r\sqrt{\omega/a} \to 0$.

The maximum values in (5) and (6) are found for arguments equal to zero. Equating the arguments to zero, from these equations we find the value of the time τ; then after calculating the derivative with respect to the variable x and applying the theorem of the inverse-function derivative, we derive the formulas by means of which we, respectively, determine the rates of temperature-wave and heat-flow wave propagation. Since

$$\frac{d\theta_n(z)}{dz} = \frac{M_{n-1}(z)}{M_n(z)}\cos\left[\theta_n(z) - \theta_{n-1}(z) - \frac{\pi}{4}\right], \tag{13}$$

$$M_{-n}(z) = M_n(z), \tag{14}$$

these formulas can be written as follows:

$$w_t = \sqrt{a\omega}\,\frac{M_0\left(r\sqrt{\frac{\omega}{a}}\right)}{M_1\left(r\sqrt{\frac{\omega}{a}}\right)}\operatorname{cosec}\left[\theta_1\left(r\sqrt{\frac{\omega}{a}}\right) - \theta_0\left(r\sqrt{\frac{\omega}{a}}\right) - \frac{\pi}{4}\right], \tag{15}$$

$$w_q = \sqrt{a\omega}\,\frac{M_1\left(r\sqrt{\frac{\omega}{a}}\right)}{M_0\left(r\sqrt{\frac{\omega}{a}}\right)}\sec\left[\theta_1\left(r\sqrt{\frac{\omega}{a}}\right) - \theta_0\left(r\sqrt{\frac{\omega}{a}}\right) - \frac{\pi}{4}\right]. \tag{16}$$

Since

$$\lim_{z\to\infty}\frac{M_0(z)}{M_1(z)} = 1, \tag{17}$$

if we consider (11), we find the following solution as $r\sqrt{\omega/a} \to \infty$:

$$w_t = w_q = \sqrt{2a\omega}. \tag{18}$$

Consequently, the rates of propagation for the temperature and heat-flow waves tend toward a value corresponding to propagation in a half-space.

Figure 1 shows the relative rates of propagation for these waves, i.e., the rate of wave propagation in a cylinder (15) and (16) as a ratio of the rate of propagation for these waves in a half-space (18).

NOTATION

t is the temperature;
a is the coefficient of thermal diffusivity;
r is the cylindrical coordinate;
τ is the time;
R is the cylinder radius;
i is an imaginary unit;
ω is the angular frequency;

λ	is the thermal conductivity;
q	is the heat flow;
n	is a natural number;
z	is the general notation for the variable;
$ber_n z$, $bei_n z$	are Thomson functions of order n;
w	is the velocity.